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Two-Body Problem with High Tangential Speeds and Quadratic Drag

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This paper considers the restricted two-body problem with atmospheric drag, and its results apply to arcs of orbits in which the tangential speed is much greater than the radial speed. In principle this treatment applies to certain arcs of any near-Keplerian orbit but its usefulness is primarily restricted to near-circular orbits. The atmospheric density is approximated by a model that enables us to derive closed-form solutions for the radial distance of a satellite in orbit that are more accurate than other closed-form solutions found in previous works. However, these new formulas for the orbit do not retain the similarity to those of the two-body problem without drag as much as the previous ones, and require the determination of two additional parameters. Nevertheless, the increase in accuracy is significant.

I. Introduction

THE problem of finding realistic closed-form solutions of the restricted two-body problem in the presence of atmospheric drag is very difficult. Some progress on this problem has been obtained through the use of perturbations and orbital elements. Some early work in this area is summarized in the paper by Lane and Cranford [1]. Later work was presented by King-Hele [2] and Hoots and France [3], and, more recently, by Vallado [4].

Another approach is to attempt to find a closed-form solution directly of the equation of motion. This requires a drag model and, most likely, certain simplifying assumptions that make the problem amenable to an analytic solution.

The equation of motion of a satellite about a spherical attractive mass subject to Newtonian gravitation and atmospheric drag is of the form

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} - \alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\dot{\boldsymbol{R}} \tag{1}$$

where $\boldsymbol{R}$ is the position vector measured from the center of mass and the upper dots represent differentiation with respect to time t; otherwise the dot represents the scalar product, $R = (\boldsymbol{R}\cdot\boldsymbol{R})^{1/2}$, $f(R) = \mu/R^3$ where μ is the product of the universal gravitational constant and the central mass, α is a constant determined from the drag coefficient, the geometry of the satellite and the atmospheric density at a specified altitude, and $\rho(R)$ is directly proportional to the atmospheric density at the radial distance R from the center of attraction.

Humi and Carter have found a way of accurately approximating and simplifying Eq. (1) for cases in which the radial component of the velocity $\dot{\boldsymbol{R}}$ is very small compared with its transverse component. With this simplification they were able to find a closed-form solution of the orbit equation that follows from Eq. (1) using the approximation

$$\rho(R) = 1/R \tag{2}$$

for the variation of the atmospheric density [5]. These approximations also enabled them to find a closed-form solution of the equations of relative motion of a spacecraft in the vicinity of a satellite [6]. Recently they also found a similar closed-form solution for the atmospheric density variation of the form

$$\rho(R) = 1/R^2 \tag{3}$$

that also leads to a closed-form solution of the orbit equation [7].

When compared with numerical integration of Eq. (1) using an exponential variation of $\rho(R)$ decreasing with R, these models were found to become increasingly inaccurate for decays in altitude beyond 1 km.

In the present paper we show that the new representation

$$\rho(R) = 1/(R-c) \tag{4}$$

leads to an improvement in accuracy over the previous models and is also amenable to a closed-form solution of the orbit equation. The reason for the increased accuracy over the proceeding models is that R and R^2 are far too large for Eqs. (2) and (3) to accurately model $\rho(R)$.

In the following pages we derive the orbit equation associated with Eq. (1), then introduce the simplification that results from the assumption of relatively small radial velocity. Introducing the approximation (4) we then obtain a linear differential equation which we solve in closed form. This solution is compared with numerical integration of Eq. (1) using an exponential variation of $\rho(R)$. The relatively large drag in the simulations is not realistic for the altitude considered. It is used only for a comparison with the results of the previous work [5,7]. We present therefore another simulation with a more realistic drag and compare its accuracy with that of King-Hele [2]. Although the present work applies in principle to a near-circular arc [7] of any orbit, its usefulness is primarily restricted to orbits that are almost circular.

II. The Orbit Equation

We introduce the angular momentum vector $\boldsymbol{L}$ by taking the vector product of Eq. (1) with $\boldsymbol{R}$ from the left, and obtain

$$\dot{\boldsymbol{L}} - \alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\boldsymbol{L} = 0 \tag{5}$$

This shows that $\dot{\boldsymbol{L}}$ is in the direction of $\boldsymbol{L}$, consequently $\boldsymbol{L}$ has a

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constant direction, and the motion under consideration takes place in a fixed plane. We can therefore introduce polar coordinates R, θ in this plane. Rewriting the equation of motion (1) in terms of polar coordinates, we have the following two equations:

$$R\ddot{\theta} + 2\dot{R}\,\dot{\theta} = -\,\alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}R\dot{\theta} \tag{6}$$

$$\ddot{R} - R\dot{\theta}^2 = -f(R)R - \alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\dot{R} \tag{7}$$

If we multiply Eq. (6) by R, divide by $R^2\dot{\theta}$, and integrate, we find that

$$R^2\dot{\theta} = J \tag{8}$$

where

$$J = e^{-\alpha\int \rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\,\mathrm{d}t} \tag{9}$$

It follows from these equations that $\dot{\theta}$ is positive unless the motion is rectilinear. Because we shall only consider cases in which the radial speed is very small compared with the transverse speed (i.e., $|\,\dot{R}\,| \ll R\dot{\theta}$), rectilinear motion is not relevant. We use Eq. (8) to change the independent variable from t to θ, and introduce this change in Eq. (7). After some algebra, the following orbit equation appears,

$$RR''(\theta) - 2R'(\theta)^2 = R^2 - f(R)R^6/J^2 \tag{10}$$

where the prime indicates differentiation with respect to θ, and J is now regarded as a function of θ. We observe that the atmospheric density $\rho(R)$ enters the orbit equation only through the expression J. We can interpret the variable J as representing a kind of "osculating specific angular momentum" (i.e., J represents the instantaneous specific angular momentum that would result at any instant if the drag could be removed at that instant). We shall show that this integro-differential Eq. (10) can be approximated, simplified, and reduced to an ordinary differential equation for cases in which $|\,\dot{R}\,| \ll R\dot{\theta}$.

III. Reduction for the Case $|\,\dot{R}\,| \ll R\dot{\theta}$

In the case where the radial speed is small compared with the tangential speed, we write

$$(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2} = (\dot{R}^2 + R^2\dot{\theta}^2)^{1/2} = R\dot{\theta}\left[1 + \left(\frac{\dot{R}}{R\dot{\theta}}\right)^2\right]^{1/2} \tag{11}$$

and linearize the right hand side, getting the approximation

$$(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2} = R\dot{\theta} \tag{12}$$

The expression (9) simplifies as a result of this approximation:

$$J = e^{-\alpha\int \rho(R)R\,\mathrm{d}\theta} \tag{13}$$

For arcs of orbits in which the radial speed is relatively small, the expression (13) can be used in the orbit Eq. (10). The reader should observe that this approximation affects the orbit equation only through the expression J.

A. Application of the New Atmospheric-Density Model

In Fig. 1, we demonstrate that the new atmospheric-density model (4) compares favorably with a model in which the density decreases exponentially with R. The value of the parameter c is determined by curve fitting procedures (e.g., method of least squares) to best approximate the atmospheric density. For an initial altitude of 7120 km we found that $c = 7005$ km provided the best fit to the data. The reader should observe that this model is not dependent on the assumption of an exponential atmospheric density. The parameter c can be fitted to any atmospheric density data. Applying this new density model (4), the expression (13) becomes

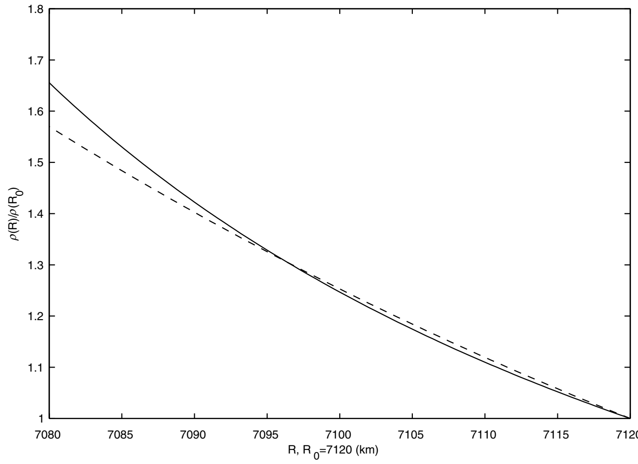

Fig. 1 Atmospheric density as a function of height normalized to 1 at 7120 km exponential model (solid line) and the new model (broken line) for $c = 7005$ km.

$$J = e^{-\alpha\int (R\,\mathrm{d}\theta)/(R-c)} \tag{14}$$

The crux of the problem is to accurately approximate Eq. (14) in a way that Eq. (10) yields a closed-form solution. We can write

$$\frac{R}{R-c} = \frac{1}{1-c/R} \tag{15}$$

but we cannot accurately approximate this expression by direct linearization because c/R is not small. To overcome this difficulty we use a trick. We let δ be a small positive number that is greater than and near $1 - c/R$, and write

$$\frac{1}{1-c/R} = \frac{1}{\delta}\left[\frac{1}{1-[\delta-(1-c/R)]/\delta}\right] \tag{16}$$

Because $|\,[\delta-(1-c/R)]/\delta\,| \ll 1$, we can now linearize the right-hand side of Eq. (16) by

$$\frac{1}{\delta}\left[1 + \frac{\delta-(1-c/R)}{\delta}\right] = \frac{2\delta-1}{\delta^2} + \frac{c/R}{\delta^2} \tag{17}$$

We now introduce the variable

$$u = \int\frac{\mathrm{d}\theta}{R} \tag{18}$$

and arrive at the approximate expression for Eq. (14),

$$J = he^{-\alpha\{[(2\delta-1)\theta/\delta^2]+(c/\delta^2)u\}} \tag{19}$$

where $h = R(0)\dot{\theta}(0)$ is the initial value of (8). Differentiating Eq. (18), substituting into the orbit Eq. (10) for Newtonian gravitation, and simplifying, we find after some manipulation, that

$$u''' + u' = \frac{\mu}{h^2}\exp\left[\frac{2\alpha(2\delta-1)\theta}{\delta^2} + \frac{2\alpha c}{\delta^2}u\right] \tag{20}$$

The constant α is very small. If the number of revolutions is not very large, we see from (18) that u is also very small, and therefore $|\,(2\alpha c/\delta^2)u\,| \ll 1$. Linearizing the exponential function in Eq. (20), we obtain the approximating linear differential equation

$$u''' + u' - \frac{2\alpha\mu c}{\delta^2 h^2}u = \frac{\mu}{h^2}\left[1 + \frac{2\alpha(2\delta-1)}{\delta^2}\theta\right] \tag{21}$$

B. Solution of the Differential Equation

Associated with Eq. (21) is the characteristic equation

$$\lambda^3 + \lambda - \frac{2\alpha\mu c}{\delta^2 h^2} = 0 \tag{22}$$

which has one positive root

$$\lambda = a > 0 \tag{23}$$

and two complex roots

$$\lambda = \sigma \pm \omega i \tag{24}$$

Solving this cubic equation, we find that

$$a = \beta - \frac{1}{3\beta} \tag{25}$$

where

$$\beta = \frac{1}{3\delta}\left[27\frac{\mu\delta\alpha c}{h^2} + 3\sqrt{3}\delta\sqrt{\delta^4 + 27\frac{\mu^2\alpha^2 c^2}{h^4}}\right]^{1/3} \tag{26}$$

$$\sigma = -a/2 \tag{27}$$

and

$$\omega = \frac{\sqrt{3}}{2}\left(\beta + \frac{1}{3\beta}\right) \tag{28}$$

The complete solution of the linear differential Eq. (21) in terms of the arbitrary constants c_1, c_2, and ϕ is as follows:

$$u(\theta) = c_1 e^{a\theta} + c_2 e^{-(a/2)\theta}\cos\omega(\theta - \phi) - \frac{\delta^2}{2\alpha c} - \frac{h^2\delta^2(2\delta - 1)}{2\alpha\mu c^2} - \frac{2\delta - 1}{c}\theta \tag{29}$$

Differentiating this expression, and applying a trigonometric identity, we find

$$u'(\theta) = ac_1 e^{a\theta} - c_2 \nu e^{-(a/2)\theta}\cos\omega(\theta - \theta_0) - \frac{2\delta - 1}{c} \tag{30}$$

where $\theta_0 = \phi + \psi$, $\cos\omega\psi = a/2\nu$, $\sin\omega\psi = \omega/\nu$ and $\nu = \frac{1}{2}(a^2 + 4\omega^2)^{\frac{1}{2}}$. Again recalling from Eq. (18) that $u' = 1/R$, we may write the solution of the orbit equation in the form

$$R = \frac{1}{ac_1 e^{a\theta} - c_2\nu e^{-(a/2)\theta}\cos\omega(\theta - \theta_0) - (2\delta - 1)/c} \tag{31}$$

We observe that the form of this solution is not reminiscent of the form of the solution of the two-body problem without drag as much as previous solutions with less accurate atmospheric-density models [5,7]. To obtain continuity at $a = 0$, one can define $c_1 = c_1'/a$ so that $\lim_{a\to 0} ac_1 = c_1'$. Note that, if $c_2 = 0$, this solution does not reduce to an exponential as do previous models [5,7].

C. Evaluation of the Arbitrary Constants

For computational purposes it is advantageous to express the arbitrary constants c_1, c_2, and θ_0 in terms of the initial conditions $R(0)$, $\dot{R}(0)$, and $\dot{\theta}(0)$ at $\theta(0) = 0$ where time t is now considered the independent variable. We shall present three equations relating these constants.

We observe from Eqs. (8) and (19), and the same approximation used to obtain the linear Eq. (21) that

$$R^2\dot{\theta} = h\left(1 - \frac{\alpha(2\delta - 1)}{\delta^2}\theta - \frac{\alpha c}{\delta^2}u\right) \tag{32}$$

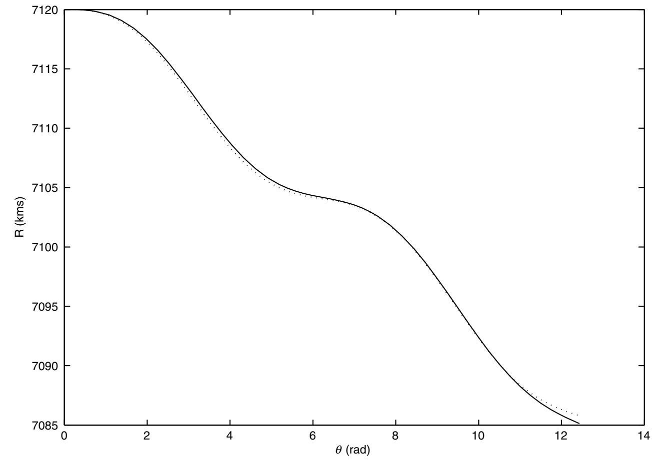

Fig. 2 Numerical solution of the equations of motion with exponential atmosphere (solid line) and closed-form solution (broken line) with $\alpha = 3.6 \times 10^{-8}$.

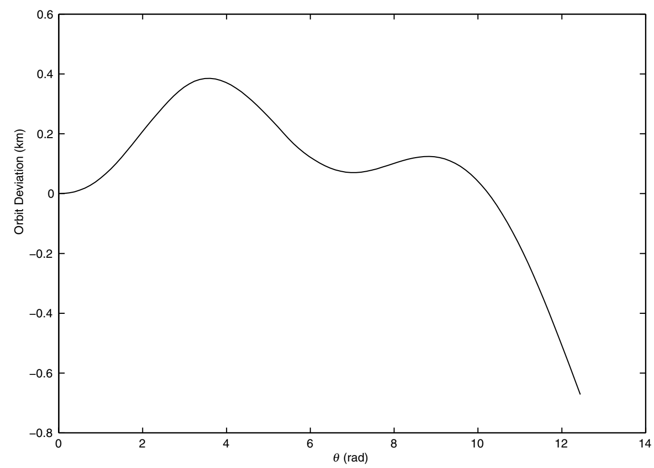

Fig. 3 Deviation between the numerical solution with exponential atmosphere and the closed-form solution with $\alpha = 3.6 \times 10^{-8}$.

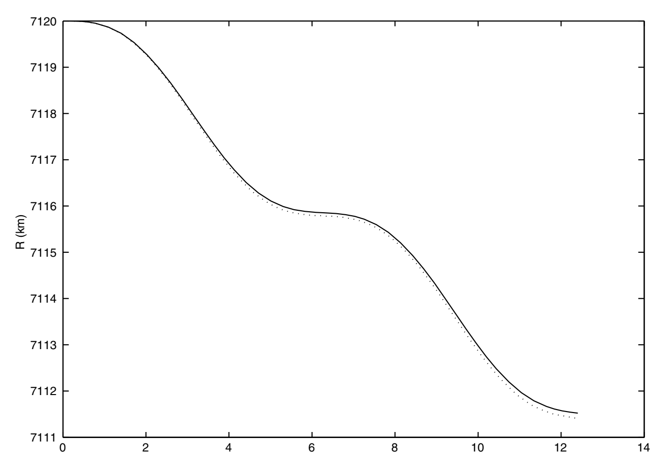

Fig. 4 Numerical solution of the equations of motion with exponential atmosphere (solid line) and closed-form solution (broken line) with $\alpha = 1.00 \times 10^{-8}$.

Combining Eqs. (29) and (32) gives the first equation. Setting $u'(\theta) = 1/R$ in Eq. (30) yields the second equation. Differentiation of the second provides the third. Using $t = 0$ and $\theta(0) = 0$, we arrive at the equations

$$c_1 + c_2\cos\omega(\theta_0 - \psi) = R_1 \tag{33}$$

$$ac_1 - \nu c_2 \cos\omega\theta_0 = R_2 \tag{34}$$

$$a^2 c_1 + \nu^2 c_2 \cos\omega(\theta_0 + \psi) = R_3 \tag{35}$$

in the three unknowns c_1, c_2, and θ_0 where

$$R_1 = \frac{h^2\delta^2(2\delta - 1)}{2\mu\alpha c^2} + \frac{\delta^2}{2\alpha c} \tag{36}$$

$$R_2 = \frac{2\delta - 1}{c} + \frac{1}{R(0)} \tag{37}$$

$$R_3 = -\frac{\dot{R}(0)}{h} \tag{38}$$

Solving Eqs. (33) and (34) for c_1 and c_2 in terms of θ_0, we obtain

$$c_1 = \frac{(2\nu^2 R_1 + aR_2)\cos\omega\theta_0 + 2\omega R_2 \sin\omega\theta_0}{(a^2 + 2\nu^2)\cos\omega\theta_0 + 2a\omega\sin\omega\theta_0} \tag{39}$$

$$c_2 = \frac{2\nu(aR_1 - R_2)}{(a^2 + 2\nu^2)\cos\omega\theta_0 + 2a\omega\sin\omega\theta_0} \tag{40}$$

Substituting these into Eq. (35), we find the angles of pseudoapse for $\theta_0 \geq 0$ from the following relation:

$$\tan\omega\theta_0 = \frac{3a^2\nu^2 R_1 + a(a^2 - \nu^2)R_2 - (a^2 + 2\nu^2)R_3}{2a\nu^2\omega R_1 - 2\omega(a^2 + \nu^2)R_2 + 2a\omega R_3} \tag{41}$$

We observe that if $\dot{R}(0) = 0$ and $\theta_0 \neq 0$ then the Eqs. (39–41) do not simplify much. However, if $aR_1 - R_2 = 0$, then $c_2 = 0$ and Eq. (31) simplifies. We remark that for previous density models [5,7] the Eqs. (33–35) and (39–41) apply, only the expressions R_1, R_2, R_3 differ.

We close with a remark on the computation of Eq. (31) from Eqs. (36–41). Because μ is large, δ is small, and α is very small, accurate computation may be troublesome. This problem is alleviated through the change of variable $\bar{\theta} = \theta/\mu^{1/3}$ and $\bar{\theta}_0 = \theta_0/\mu^{1/3}$ in the differential Eq. (21), and the appropriate use of the chain rule. A solution analogous to Eq. (31) in terms of $\bar{\theta}$ and $\bar{\theta}_0$ instead of θ and θ_0 may be preferred for computational purposes.

D. Simulations and Comparisons

A comparison of Eq. (31) and numerical integration of the orbit equation with exponential atmospheric density for $R(0) = 7120$ km, $\dot{R}(0) = 0$, $c = 7005$ km, $\delta = 0.01604$, $\alpha = 3.6 \times 10^{-8}$, and $\rho_0 = 2.27 \times 10^{-10}$ kg/km^3 is presented in Fig. 2. For a loss in altitude of 35 km the orbit determined analytically (indicated by the broken curve) accurately follows the path defined by numerical integration of the equations of motion with an exponential atmospheric-density model (indicated by the solid curve). The actual deviation from the path calculated numerically can be seen from Fig. 3.

This calculation is presented for demonstration purposes only. A satellite at this altitude would require an enormous cross-sectional area to drop 35 km in two revolutions. It is presented here for comparison with previous models [5,7] to show the vast improvement in accuracy. Under the same conditions, the new model displays the same level of accuracy for the 35 km decay as these earlier models showed for a decay of only 1 km.

A somewhat more realistic example is presented in Fig. 4. In this example all the conditions are the same as before but the cross-sectional area is lower so that $\alpha = 1.00 \times 10^{-8}$. This results in a drop of 4.3 km for the first revolution and accurately follows the numerical simulation (indicated by the solid curve) for two revolutions. Figure 5 shows the accuracy of Eq. (31) when compared with the numerical simulation of the equations of motion.

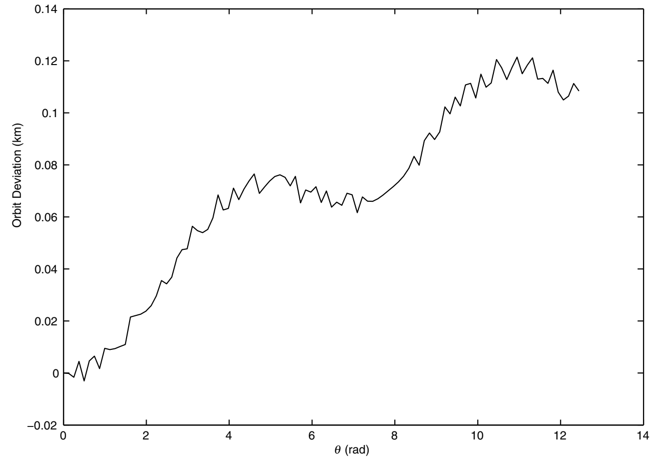

Fig. 5 Deviation between the numerical solution with exponential atmosphere and the closed-form solution with $\alpha = 1.00 \times 10^{-8}$.

The result of the calculation of Eq. (31) is compared with the analytical result given in the King-Hele book [2] by formula 4.84. Using the same data the King-Hele formula gives a drop of 3.6 km per revolution. This comparison shows that Eq. (31) gives improvement in accuracy of more than half a kilometer in one revolution for this example.

IV. Conclusion

There is no exact analytical formula that represents the atmospheric density in terms of altitude, although this dependence has sometimes been approximated locally by an exponential function that decreases with altitude. This work presents an approximate atmospheric-density model that leads to a closed-form solution for the radial distance of a satellite in the presence of drag that is more accurate than previous closed-form solutions to this problem. The resulting formula, unfortunately loses some of its similarity to that of the two-body problem without drag and is somewhat more cumbersome than the earlier less-accurate solutions.

Comparison with numerical integration using an exponential atmospheric-density model shows that the new formula is more accurate than previous ones found in the literature.

Usefulness of this work is limited to nearly circular orbits and is restricted to only a few revolutions, but is applicable to altitudes and configurations with relatively large drag. Possible applications include its use in quick closed-form calculations for feasibility studies and validation in mission planning, or as a teaching tool to demonstrate the effect of various parameters on the altitude of a satellite under the influence of drag.

References

[1] Lane, M. H., and Cranford, K. H., "An Improved Analytical Drag Theory for the Artificial Satellite Problem," AIAA Paper 69-925, Aug. 1969.

[2] King-Hele, D., *Satellite Orbits in an Atmosphere*, Blackie, London, 1987.

[3] Hoots, F. R., and France, R. G., "An Analytic Satellite Theory Using Gravity and Dynamic Atmosphere," *Celestial Mechanics*, Vol. 40, No. 1, 1987, pp. 1–18.

[4] Vallado, D. A., *Fundamentals of Astrodynamics and Applications*, McGraw–Hill, New York, 1997.

[5] Humi, M., and Carter, T., "Models of Motion in a Central-Force Field with Quadratic Drag," *Celestial Mechanics and Dynamical Astronomy*, Vol. 84, No. 3, 2002, pp. 245–262.

[6] Carter, T., and Humi, M., "Clohessy-Wiltshire Equations Modified to Include Quadratic Drag," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 6, Nov.–Dec. 2002, pp. 1058–1063.

[7] Humi, M., and Carter, T., "Closed-Form Solutions for Near-Circular Arcs with Quadratic Drag," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, May–June 2006, pp. 513–518.